Math 656 • FINAL EXAM • May 8, 2014

1) (18pts) Find and categorize **all zeros and singularities** of the following functions (you don't have to examine possible singularity at z=∞); make sure to explain briefly:

$$f(z)=\frac{\sin(1/z)}{\left(1+e^z\right)^2} \Rightarrow \begin{cases} \text{Simple zeros: } \sin(1/z)=0 \Rightarrow \frac{1}{z}=\pi k \ (k=\pm1,\pm2,\pm3) \Rightarrow \boxed{z_k=\frac{1}{\pi k}} \\ \text{Essential singularity: } \boxed{z_o=0} \\ \text{Poles of order } m=2\text{: } 1+e^z=0 \Rightarrow e^z=-1 \Rightarrow z_k=i\pi+i2\pi k \Rightarrow \boxed{z_k=i\pi(2k+1)} \end{cases}$$

$$f(z)=z^2\tanh\frac{1}{z}=\frac{z^2\sinh\frac{1}{z}}{\cosh\frac{1}{z}} \Rightarrow \begin{cases} \text{Simple zeros: } \sinh(1/z)=0 \Rightarrow \frac{1}{z}=i\pi k \ (k\in\mathbb{Z},\ k\neq0) \Rightarrow \boxed{z_k=\frac{i}{\pi k}} \\ \text{Simple poles: } \cosh(1/z)=0 \Rightarrow \frac{1}{z}=i\pi\left(k+\frac{1}{2}\right) (k\in\mathbb{Z}) \Rightarrow \boxed{z_k=\frac{i}{\pi\left(k+\frac{1}{2}\right)}} \\ \text{Cluster point: } \boxed{z_o=0} \end{cases}$$

$$f(z)=\frac{z^{1/3}-1}{e^z-e} \Rightarrow \begin{cases} \text{Branch point: } \boxed{z_o=0} \\ \text{Simple poles: } e^z=e \Rightarrow e^{z-1}=1 \Rightarrow \boxed{z_k=1+i2\pi k} \ (k\in\mathbb{Z},\ k\neq0) \\ \text{Removable singularity: } \boxed{z_o=1} \end{cases}$$

Explanation for the removable singularity at $z=1$:

$$\zeta\equiv z-1 \Rightarrow f(z)=\frac{(\zeta+1)^{1/3}-1}{e^{\zeta+1}-e}=\frac{(\zeta+1)^{1/3}-1}{e\left(e^{\zeta}-1\right)}=\frac{\left(1+\frac{\zeta}{3}+O\left(\zeta^2\right)\right)-1}{e\left(\zeta+O(\zeta^2)\right)}=\boxed{\frac{1}{3e}+O(\zeta)}$$

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2) (18pts) Find the series representation of the following functions in the indicated regions:

(a) $f(z) = \dfrac{z e^z}{\sin^2 z}$ in $0 < |z| < \pi$ (find the **first 3 dominant terms only**)

$$f(z) = \frac{z e^z}{\sin^2 z} = \frac{z\left(1 + z + \frac{z^2}{2} + O(z^3)\right)}{\left(z - \frac{z^3}{3!} + O(z^5)\right)^2} = \frac{1 + z + \frac{z^2}{2} + O(z^3)}{z\left(1 - \frac{z^2}{3!} + O(z^5)\right)^2} = \frac{1 + z + \frac{z^2}{2} + O(z^3)}{z\left(1 - \frac{2z^2}{3!} + O(z^4)\right)}$$

$$= \frac{1}{z}\left(1 + z + \frac{z^2}{2} + O(z^3)\right)\left(1 + \frac{z^2}{3} + O(z^4)\right) = \frac{1}{z}\left(1 + z + \frac{z^2}{2} + \frac{z^2}{3} + O\left(z^3\right)\right) = \boxed{\frac{1}{z} + 1 + \frac{5}{6}z + O\left(z^2\right)}$$

(b) $f(z) = \dfrac{z}{z^2 - 1}$ in $1 < |z-2| < 3$ (use partial fraction decomposition and the geometric series)

$$\left.\begin{array}{l} \zeta \equiv z - 2 \\ z \equiv \zeta + 2 \end{array}\right\} \quad 1 < |z-2| < 3 \quad \Rightarrow \quad 1 < |\zeta| < 3 \quad \Rightarrow \left\{\begin{array}{l} \dfrac{1}{|\zeta|} < 1 \\ \dfrac{|\zeta|}{3} < 1 \end{array}\right.$$

$$\Rightarrow \ f(z) = \frac{z}{z^2 - 1} = \frac{1}{2}\left(\frac{1}{z+1} + \frac{1}{z-1}\right) = \frac{1}{2}\left(\frac{1}{\zeta + 3} + \frac{1}{\zeta + 1}\right) = \frac{1}{6}\frac{1}{1 + \frac{\zeta}{3}} + \frac{1}{2\zeta}\frac{1}{1 + \frac{1}{\zeta}}$$

$$= \frac{1}{6}\sum_{k=0}^{\infty}\left(-\frac{\zeta}{3}\right)^k + \frac{1}{2\zeta}\sum_{k=0}^{\infty}\left(-\frac{1}{\zeta}\right)^k = \boxed{\frac{1}{6}\sum_{k=0}^{\infty}\frac{(-1)^k}{3^k}(z-2)^k - \frac{1}{2}\sum_{k=1}^{\infty}\frac{(-1)^k}{(z-2)^k}}$$

3) (24pts) Calculate the following integrals. Carefully explain each step, and make sure to obtain a **real** answer:

$$(a)\quad \int_{-\infty}^{\infty}\frac{dx}{\left(x^2+a^2\right)^2} \Rightarrow \oint\frac{dz}{\left(z^2+a^2\right)^2} = \int_{-R}^{R}\frac{dx}{\left(x^2+a^2\right)^2} + \underbrace{\int_{C_R}\frac{dz}{\left(z^2+a^2\right)^2}}_{|\ldots|\le\frac{\pi R}{\left(R^2-a^2\right)^2}\to\frac{\pi}{R^3}\to 0} = 2\pi i\ \mathrm{Res}(ia)$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{dx}{\left(x^2+a^2\right)^2} = 2\pi i\ \mathrm{Res}(ia) = 2\pi i\left.\frac{d}{dz}\frac{1}{\left(z+ia\right)^3}\right|_{z=ia} = -\frac{4\pi i}{\left(2i\,a\right)^3} = \boxed{\frac{\pi}{2a^3}}$$

$$(b)\quad \oint\frac{e^{iz}\,dz}{\underbrace{z^2-2z+2}_{(z-1-i)(z-1+i)}} = \int_{-R}^{R}\frac{e^{ix}\,dx}{x^2-2x+2} + \underbrace{\underbrace{\int_{C_R}\frac{e^{iz}\,dz}{z^2-2z+2}}\quad |\ldots|\le\frac{\pi R}{\left(R-\sqrt{2}\right)^2}\to\frac{\pi}{R^3}\to 0} = 2\pi i\ \mathrm{Res}(1+i) = 2\pi i\frac{e^{i(1+i)}}{2i} = \pi e^{-1}\left(\cos 1 + i\sin 1\right)$$

We don't really need the Jordan's Lemma here; if we did use it, we'd obtain an extra power of R in denominator

$$\Rightarrow \int_{-\infty}^{\infty}\frac{\cos x\,dx}{x^2-2x+2} = \mathrm{Re}\int_{-\infty}^{+\infty}\frac{e^{ix}\,dx}{x^2-2x+2} = \boxed{\pi e^{-1}\cos 1}$$

(c) $\int_0^\infty \frac{x^p\,dx}{x^4+1} \Rightarrow \oint \frac{z^p\,dz}{z^4+1} = \underbrace{\int_0^R \frac{r^p\,dr}{r^4+1}}_{I} - \underbrace{\int_0^R \frac{(ir)^p\,d(ir)}{(ir)^4+1}}_{i^{p+1}I} + \int_{C_\varepsilon} \frac{z^p\,dz}{z^4+1} + \int_{C_R} \frac{z^p\,dz}{z^4+1} = 2\pi i \operatorname{Res}\left(z_1 = e^{i\pi/4}\right)$

$$= 2\pi i \frac{z_1^p}{4z_1^3} = -\frac{\pi i}{2} z_1^{p+1} = \frac{-\pi i}{2} e^{i\pi(p+1)/4}$$

$$\left.\begin{aligned} \left|\int_{C_\varepsilon} \frac{z^p\,dz}{z^4+1}\right| &\le \frac{\varepsilon^p \pi\varepsilon}{1-\varepsilon^4} \to \pi\varepsilon^{p+1} \to 0 \text{ as } \varepsilon \to 0 \text{ if } p+1>0 \Rightarrow p>-1 \\ \left|\int_{C_R} \frac{z^p\,dz}{z^4+1}\right| &\le \frac{R^p \pi R}{R^4-1} \to \frac{\pi}{R^{3-p}} \to 0 \text{ as } R\to\infty \text{ if } 3-p>0 \Rightarrow p<3 \end{aligned}\right\} \Rightarrow \boxed{-1<p<3}$$

Take the limit $\varepsilon \to 0$ and $R \to \infty$:

$$\left(1-i^{p+1}\right)I = \left(1-e^{i\pi(p+1)/2}\right)I = \frac{-\pi i}{2} e^{i\pi(p+1)/4} \Rightarrow I = -\frac{i\pi}{2} \frac{e^{i\pi(p+1)/4}}{1-e^{i\pi(p+1)/2}} \cdot \frac{e^{-i\pi(p+1)/4}}{e^{-i\pi(p+1)/4}} = \frac{i\pi}{2} \frac{1}{e^{i\pi(p+1)/4} - e^{-i\pi(p+1)/4}} = \boxed{\frac{\pi}{4\sin\frac{\pi(p+1)}{4}}}$$

In (c), find also the convergence condition on real constant p

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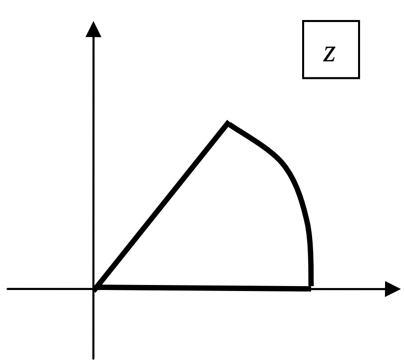
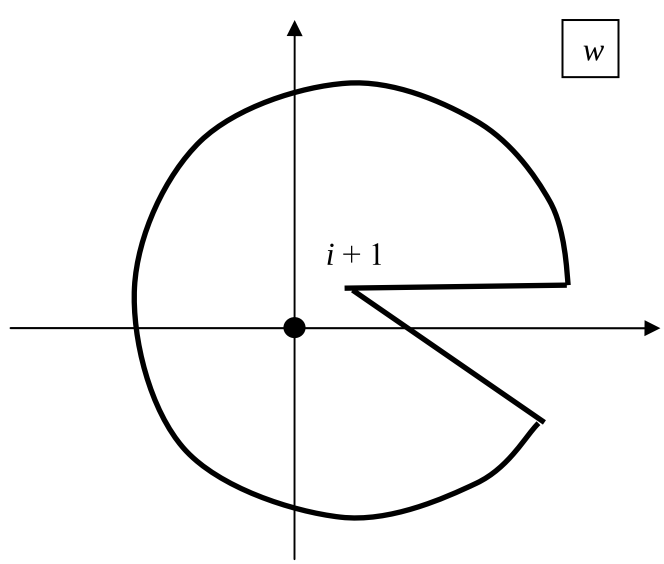
4) **(12pts)** Use the Argument Principle to find the number of zeros of function $f(z) = z^7 + i + 2$ located within the sector $0 \le \arg z < \frac{\pi}{4}$

Answer: N=1= winding number of the image of the contour enclosing this sector:





5) (12pts) Show that the transformation $w = \dfrac{z-\alpha}{1-\bar{\alpha}z}$ (where α is a complex constant satisfying $|\alpha|<1$) maps a unit disk into itself (hint: examine the mapping of the **unit circle** by calculating $|w|^2$).

$$|w|^2 = w\bar{w} = \frac{z-\alpha}{1-\bar{\alpha}z}\cdot\frac{\bar{z}-\bar{\alpha}}{1-\alpha\bar{z}} = \frac{|z|^2-\alpha\bar{z}-\bar{\alpha}z+|\alpha|^2}{1-\alpha\bar{z}-\bar{\alpha}z+|\alpha|^2|z|^2}$$

On the unit cirlce $|z|^2=1$, therefore $|w|^2 = \dfrac{1-\alpha\bar{z}-\bar{\alpha}z+|\alpha|^2}{1-\alpha\bar{z}-\bar{\alpha}z+|\alpha|^2} = 1$

Now we only have to prove that a point insize the unit disk maps inside the unit disk: take $z=\alpha$:

$w(\alpha)=0$

---------------------------------- **Choose between problems 6 and 7** ---

6) (16pts) Find the coefficients C_{-1}, C_{-2} and C_{-3} in the **principle part** of the Laurent series for $f(z)=\dfrac{z}{\sin z}$ converging within $\pi<|z|<2\pi$. This will help you in finding **all** coefficients in the principle part.

$$C_n = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{z^{n+1}} \Rightarrow \text{For the principle part we have } C_{-n} = \frac{1}{2\pi i}\oint_C \frac{f(z)\,dz}{z^{-n+1}} = \frac{1}{2\pi i}\oint_C z^{n-1} f(z)\,dz$$

$$C_{-1} = \frac{1}{2\pi i}\oint_C f(z)\,dz = \frac{1}{2\pi i}\oint_C \frac{z}{\sin z}dz = \text{Res}(\pi)+\text{Res}(-\pi) = \frac{\pi}{\cos\pi}+\frac{-\pi}{\cos(-\pi)} = \boxed{0}$$

$$C_{-2} = \frac{1}{2\pi i}\oint_C z\,f(z)\,dz = \frac{1}{2\pi i}\oint_C \frac{z^2}{\sin z}dz = \text{Res}(\pi)+\text{Res}(-\pi) = \frac{\pi^2}{\cos\pi}+\frac{(-\pi)^2}{\cos(-\pi)} = \boxed{-2\pi^2}$$

$$C_{-3} = \frac{1}{2\pi i}\oint_C z^2\,f(z)\,dz = \frac{1}{2\pi i}\oint_C \frac{z^3}{\sin z}dz = \text{Res}(\pi)+\text{Res}(-\pi) = \frac{\pi^3}{\cos\pi}+\frac{(-\pi)^3}{\cos(-\pi)} = \boxed{0}$$

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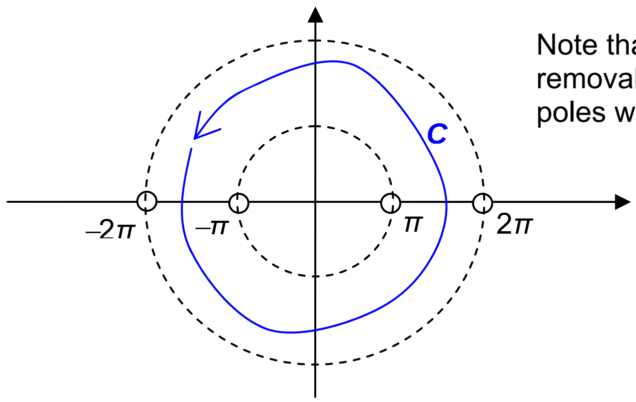
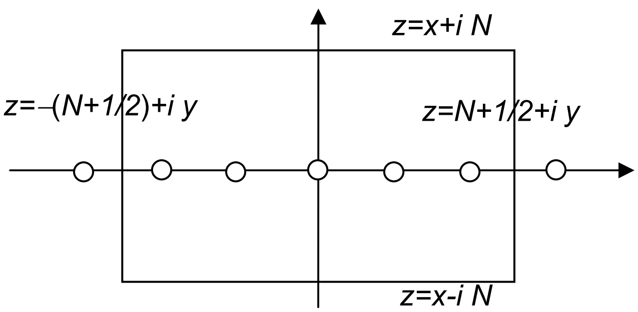
$$C_{-n} = \frac{1}{2\pi i}\oint_C z^{n-1} f(z)\,dz = \frac{1}{2\pi i}\oint_C \frac{z^n}{\sin z}dz = \text{Res}(\pi)+\text{Res}(-\pi) = \frac{\pi^n}{\cos\pi}+\frac{(-\pi)^n}{\cos(-\pi)} = \boxed{\begin{cases}0, & n \text{ is odd}\\ -2\pi^n, & n \text{ is even}\end{cases}}$$



Note that the singularity at z=0 is removable, so there are only 2 simple poles within the integration contour

7) (16pts) Calculate $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ by integrating $f(z)=\frac{\pi}{z^2\sin(\pi z)}$ over a rectangular contour with sides formed by lines $\pm\left(N+\frac{1}{2}\right)+i\,y$ and $x\pm iN$ (where N is an integer) and then taking the limit $N\to\infty$



Along horizontal segments 1 / |sin(πz)| ≤ 1 / |exp(πN)-exp(-πN)|, exponentially decaying as N = | Im z | →∞, therefore the integral also approaches zero in this limit. Along the vertical segments z=±(N+1/2)+i y, we have

$$\frac{1}{|\sin\pi z|}=\frac{1}{\left|\sin\left(\pm\left(N+\tfrac{1}{2}\right)\pi+i\pi y\right)\right|}=\frac{1}{\left|\sin\left(\pm\left(N+\tfrac{1}{2}\right)\pi\right)\cosh\pi y+i\cos\left(\pm\left(N+\tfrac{1}{2}\right)\pi\right)\sinh\pi y\right|}=$$

$$=\frac{1}{\left|\sin\left(\pm\left(N+\tfrac{1}{2}\right)\pi\right)\cosh\pi y\right|}=\frac{1}{\cosh\pi y}\le 1$$

So the integral is bounded by the length of the contour, 2N, times $1/|z|^2\sim 1/N^2$, yielding zero in the limit $N\to\infty$. Thus, the sum of residues is zero:

$$\sum_{n=-\infty}^{\infty}\mathrm{Res}\left(f(z),z_n=n\right)=\mathrm{Res}\left(f(z),0\right)+2\sum_{n=1}^{\infty}\mathrm{Res}\left(f(z),n\right)=\mathrm{Res}\left(f(z),0\right)+2\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\Rightarrow\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=-\frac{\mathrm{Res}\left(f(z),0\right)}{2}$$

Now let's find the residue at 0:

$$\frac{\pi}{z^2\sin\pi z}=\frac{\pi}{z^2\left(\pi z-\frac{(\pi z)^3}{3!}+O(z^5)\right)}=\frac{1}{z^3}\frac{1}{1-\frac{(\pi z)^2}{6}+O(z^4)}=\frac{1}{z^3}\left(1+\frac{(\pi z)^2}{6}+O(z^4)\right)=\frac{1}{z^3}+\frac{\pi^2}{6}\frac{1}{z}+O(z)$$

$$\Rightarrow\quad \boxed{\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}=-\frac{\pi^2}{12}}$$

The answer is negative since the first (dominant) term is negative.